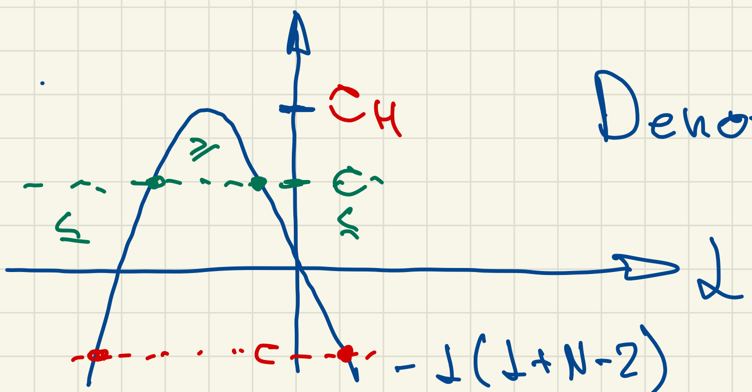
TCC Week 6.

Example. $-\Delta u - \frac{c}{|x|^2} u = 0$ in $\overline{B}_1^c$

Take $u = r^\alpha$ as an "ansatz".

$$-\Delta r^\alpha - \frac{c}{r^2} r^\alpha = -\alpha(\alpha+N-2) r^{\alpha-2} - c r^{\alpha-2}$$

$$= \left(-\alpha(\alpha+N-2) - c\right) r^{\alpha-2} \begin{cases} \leq 0 & \text{if } \alpha \geq \alpha_+ \\ \geq 0 & \text{if } \alpha_- \leq \alpha \leq \alpha_+ \\ \leq 0 & \text{if } \alpha \leq \alpha_- \end{cases}$$



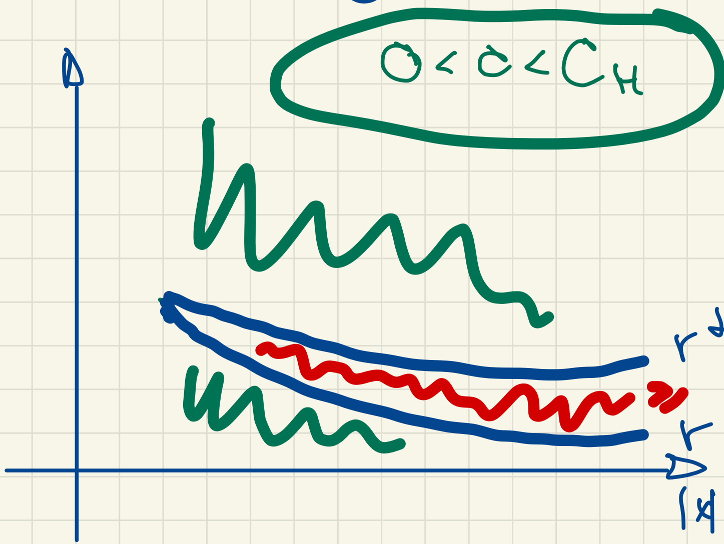
Denote $\alpha^- < \alpha^+$ roots

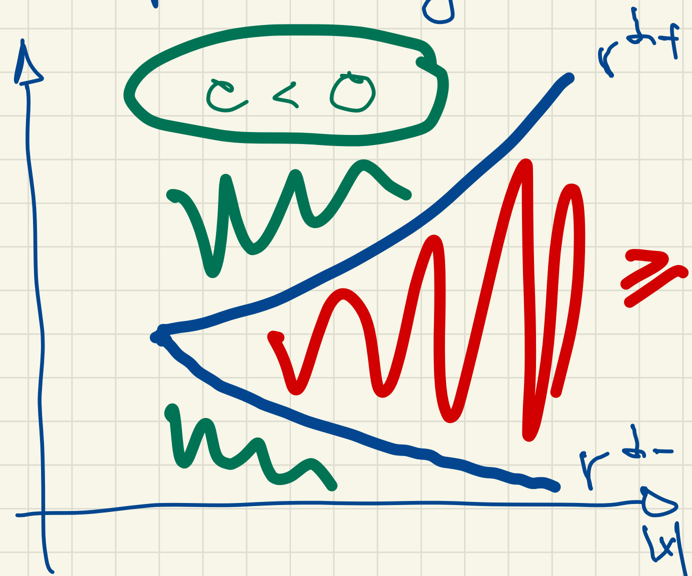
$$-\alpha(\alpha+N-2) = c,$$

$$c < c_H = \frac{(N-2)^2}{4}$$

r^{α_-} is minimal solution = optimal small subsol

$r^{\alpha_+} - r^{\alpha_-}$ is Agmon's solution = optimal large subsol





Nonlinear Liouville theorems

Consider nonlinear equation

(*) $-\Delta u = u^p$ in $\Omega = \mathbb{R}^N \setminus \overline{B}_1$, $N \geq 3$, $p \in \mathbb{R}$

$0 \leq u \in H^1_{loc}(\Omega)$ is a nonnegative weak supersol

if $\int_\Omega \nabla u \nabla \varphi \geq \int_\Omega u^p \varphi \quad \forall \varphi \in C_c^\infty(\Omega), \varphi \geq 0$

Remarks: 1) $u^p \in L^1_{loc}(\Omega)$

2) $-\Delta u \geq u^p$, $u \geq 0 \Rightarrow -\Delta u \geq 0$

$\Rightarrow$ u is superharmonic

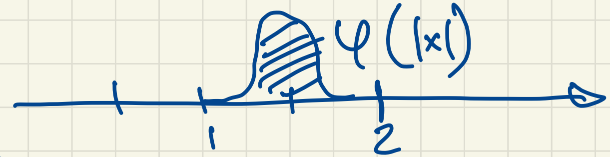
Theorem. (Serrin $\simeq$ 76')

(*) has positive supersol $\iff$ $p > \underbrace{\frac{N}{N-2}}_{\text{Serrin critical exponent}}$

▲ Difficult part is nonexistence.

o) Trivial case $p=1$ Take a $\varphi \in C_c^\infty(\Omega)$

$$\varphi_R(x) = \varphi\left(\frac{|x|}{R}\right)$$



$$\int |\nabla \varphi_R|^2 - \int |\varphi_R|^2 = R^{N-2}\int |\nabla\varphi|^2 - R^N \int \varphi^2 \xrightarrow[R\to+\infty]{} -\infty$$

By AAP $-\Delta u \geq u$ has no positive supersolutions in Ω.

1) Assume $p > 1$ (superlinear case)

Assume $\exists\, u > 0$: $-\Delta u \geq u^p$ in Ω.

Then $-\Delta u \geq 0$ in Ω

$\Longrightarrow$ $u \geq c|x|^{2-N}$ in $\overline{B}_R^c$, $R > 1$.

Linearisation step:

$$-\Delta u \geq (u^{p-1})\, u \geq c^{p-1}|x|^{(2-N)(p-1)}\, u \quad \text{in } \Omega$$

$V(x)$ ~ linearisation potential

? $(2-N)(p-1) > -2 \iff 1 < p < \frac{N}{N-2}$

$$-\Delta u \geq |x|^{\gamma} u \quad \text{in } \Omega, \qquad (\gamma = (2-N)(p-1))$$

$\gamma > -2$ if $p < \frac{N}{N-2}$

$\Rightarrow$ $u \equiv 0$ by scaling argument and AAP principle, just as before!

Critical case $p = \frac{N}{N-2}$?

$$-\Delta u \geq c|x|^{-2} u \quad \text{in } \Omega$$ — Hardy potential!

If we were to know that $c > c_H \Rightarrow$

$\Rightarrow u \equiv 0$ = nonexistence.

Assume $0 < c \leq c_H$. Then

$$-\Delta u \geq \frac{c}{|x|^2} u \quad \text{in } \Omega$$

Then $u \geq c|x|^{\alpha_-}$ in B_2^c

and $2-N < \alpha_- < 0$. Now repeat linearisation trick with "improved" estimate:

$$-\Delta u \geq \underbrace{(u^{p-1})}_{V(x)} u \quad , \quad V(x) = u^{p-1} \geq c_1 |x|^{(p-1)\alpha_-} \quad ,$$

$$(p-1)\alpha_- > -2 \implies \text{nonexistence } (u \equiv 0)$$

$p<1$

$$-\Delta u - \underbrace{u^{p-1}}_{V(x)} u \geq 0$$

$p-1<0 \Rightarrow$ lower bound on u gives upper bound on $V(x)$

Upper bound from large subsolutions is not a pointwise bound.

We can say $-\Delta u \geq 0 \Rightarrow \liminf u > c$

$\Rightarrow \limsup V(x) < c$?!

Linearisation argumen fails.

Lemma A $-\Delta u \geq u^p$ in Ω, $p<1$

$\Rightarrow u \geq c|x|^{\frac{2}{1-p}}$ in B_2^c

uses AAP + weak Harnack

Exercise: check that

$$U_*(x) = c|x|^{\frac{2}{1-p}} \text{ is a solution to}$$

$$-\Delta U_* = U_*^p \quad \text{for some } c>0$$

Hint $U_*^{p-1} = c^{p-1}|x|^{-2}$

Case $p<1$ – nonexistence

By Lemma, $u \geq c|x|^{\frac{2}{p-1}} \to +\infty$ as $|x| \to \infty$

By Phragmen-Lindelöf, $\liminf_{|x|\to\infty} u < +\infty$

– a contradiction!

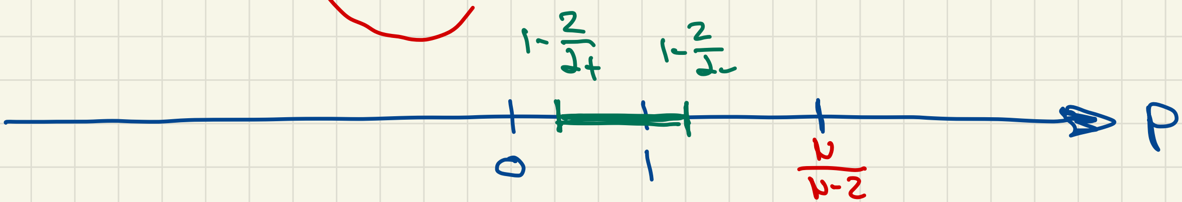
$\Rightarrow u \equiv 0$ ◪

Exercise: $-\Delta u - \frac{c}{|x|^2} u \geq u^p$ in Ω,

$c \leq c_H$ (otherwise $-\Delta u - \frac{c}{|x|^2} u \geq 0 \Rightarrow u \equiv 0$

has a positive super-solution $\Longleftrightarrow$

$p \notin \left[1 - \frac{2}{\lambda_+}, 1 - \frac{2}{\lambda_-}\right]$ $\qquad 1 - \frac{2}{\lambda_+}$ "=" $-\infty$ if $c > 0$



1) $c = 0 \Longrightarrow p \notin \left[-\infty, \frac{N}{N-2}\right]$

2) $c = c_H \Longrightarrow p \notin \left[-\infty, \frac{2N}{N-2}\right]$

3) $c < 0 \Longrightarrow p \notin \left[1 - \frac{2}{\lambda_+}, 1 - \frac{2}{\lambda_-}\right]$

Hint. Assume $p < 1$:

Lemma A remains valid with no changes:

$$-\Delta u - \frac{c}{|x|^2} u \geqslant u^p \implies u \geq c_1 |x|^{\frac{2}{1-p}}$$

By Phragmen-Lindelöf

$$\liminf \frac{u}{|x|^{\alpha_+}} < +\infty$$

$$\alpha_+ < \frac{2}{1-p} \iff p > 1 - \frac{2}{\alpha_+}$$

Existence is trivial:

$u = c r^{\gamma}$ is a supersolution for a suitable γ in the existence regions!

$$-\Delta u + u^p = 0 \quad \text{in } \Omega$$